Mathematical tools for secondary school

Michele Tuttafesta

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Advanced topics

Preface

Dear Readers, “Mathematical Tools for Secondary School” might evoke images of piles of numbers, equations, and theorems ready to overwhelm you. But trust me, this book is more like a treasure map, designed to guide you through the vast world of mathematics with confidence and, why not, even a bit of fun.

This book doesn’t aim to convince you that math is useful in life or that it can be beautiful (though it certainly is!). It assumes you’re already motivated: whether you’re a student eager to understand better, a teacher looking for new tools, or simply an enthusiast ready to dive into this sea of formulas and ideas. The purpose is simple: to offer you a practical and comprehensive study companion, something to consult when you have a question or can’t recall a definition, a property, a noteworthy result, or when you need an idea to solve a problem. Here you’ll find a bit of everything: well-organized statements, accessible proofs, meaningful or intriguing problems, and insights for those who want to go beyond the standard curriculum.

For students, this book will be an ideal toolbox for review, daily study, or the dreaded final exam. For teachers, it will be a handy reference tool (thanks to the detailed index) to build clear, complete, and focused lesson plans, avoiding distractions and centering on the core of each topic.

I would like to thank everyone who contributed to my mathematical education. Elementary School: Giuseppe Chiauzzi (Arithmetic and Geometry). Middle School: Antonio Luisi (Arithmetic, Algebra, and Geometry). University: Irene Sisto (Mathematical Analysis), Biagio Casciaro (Linear Algebra and Geometry), Nicola Cufaro and Adriano Massaro (Complex Analysis), Gianpiero Colonna (Numerical Calculus).

Now, make yourselves comfortable (but not too much—mathematics demands concentration!), sharpen your pencils or activate your digital tools, and try to discover what you thought mathematics couldn’t do. You can do it, and you’ll probably enjoy it.

Michele Tuttafesta

Introduction

The basic concepts expressed in this book are systematically presented as statements (formal expressions that can be evaluated as true or false), grouped into the following categories:

Properties	statements verifiable by a formal proof.
Theorems	property of particular relevance or generality.
Postulates	statements whose truth is in some way evident; therefore they do not require demonstration.
Definitions	specific and unambiguous formal agreements; generally do not require any demonstration.
Rules	statements consisting of finite sequences of instructions, programmed to achieve a certain result; they may require demonstrations that justify certain sequences.

Many properties are demonstrated in detail. Some results (very few, in fact) are provided without proof, either because they are easy to prove or because they require knowledge beyond the intended scope of this text.

In the first chapter, the “mathematical tools” are divided into sections with self-explanatory titles: Arithmetic, Algebra, Geometry..., Vectors, Trigonometry, Calculus, etc.

The second chapter contains a brief introduction to the fantastic LaTeX, a word-processing system of unparalleled elegance and power, designed for advanced typesetting and universally acclaimed for its ability to produce documents of impeccable editorial quality. Created in 1984 by Leslie Lamport as an extension of the TeX typesetting system conceived by Donald Knuth, LaTeX has become the de facto standard for drafting scientific articles. Its importance in the mathematical and scientific community is unmatched, thanks to its capability to handle complex mathematical formulas with unparalleled clarity and precision. In an era dominated by simplification and standardized digital aesthetics, LaTeX remains a beacon of intellectual rigor and typographic beauty.

The third chapter delves into topics that, at the secondary school level, may interest particularly curious students or teachers intending to propose enrichment activities.

The fourth, brief, chapter concludes the text with a collection of formulas and results of notable scientific interest.

At the end of many sections, problems are proposed, almost always with data to be chosen freely, followed by explanatory solutions. These problems are designed to be significant to the topics covered, even though they are not numerous overall. A richer collection of problems and examples, as well as alternative theoretical treatments compared to those presented in this book, can be found in the excellent texts by Domenico D'Ortenzi ([1],[2],[3]).

Chapter 1

Mathematical tools

Oh students, study mathematics, and do not build without foundations.
Mechanics is the paradise of mathematics because here its fruits can be harvested. There is no certainty in science if mathematics cannot be applied to it or if it is not related to it in some way.

Leonardo da Vinci (1452-1519): inventor, artist, scientist, man of ingenuity and universal talent.

Primitive concepts: set, point, line, plane, space, length, surface, rigid movement.

1.1 Arithmetic

Definition 1.1 (number sets) .

Name	*Symbol and specification*
Natural	$\mathbb{N} = \{0, 1, 2, 3, \dots\}$
Integer	$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$
Rational	$\mathbb{Q} = \left\{ \frac{n}{d} \middle\vert\, n, d \in \mathbb{Z}, d \neq 0 \right\} = \{\textit{All the fractions.}\}$
Irrational	$\mathbb{I} = \{\textit{Numbers that have an unlimited and non-periodical representation.}\}$
Real	$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$
Complex	$\mathbb{C} = \{a + i \cdot b \vert\, a, b \in \mathbb{R}, i^2 = -1\}$

It's obvious that: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Definition 1.2 (divisibility between integers) *Let n and d be two integers. We say that d divides n, or that n is divisible by d (denoted as $d|n$), if there exists an integer q such that $n = q \cdot d$.*

Property 1.1 (divisibility criteria) .

d	*Criterion for an integer n to be divisible by d*
2	*The units digit must be even, that is, a multiple of 2.*
3	*The sum of the digits must be a multiple of 3.*
5	*The units digit must be 0 or 5.*
7	*The absolute value of the difference between the number written without the units digit and twice the units digit must be a multiple of 7.*
11	*The absolute value of the difference between the sum of the digits in the even places and the sum of the digits in the odd places must be a multiple of 11.*
13	*The sum of the number without the units digit and four times the units digit must be a multiple of 13.*
17	*The absolute value of the difference between the number without the units digit and five times the units digit must be a multiple of 17.*
23	*The sum of the number without the units digit and 7 times the units digit must be a multiple of 23.*
37	*The number is divided into "triples". If the sum of all the triplets is a multiple of 37, then the starting number is divisible by 37. Alternatively, you can calculate the remainders of the divisions by 37 of the various triplets and finally add all the remainders obtained: if this sum gives as a result a multiple of 37 then the starting number is divisible by 37.*

Proof. See Section 3.1 and Property 3.3.
Examples.
• The number 1003 is divisible by 17 if the number $|100 - 5 \cdot 3| = 85$ is divisible by 17. This number is divisible by 17 if the number $|8-5 \cdot 5| = 17$ is divisible by 17.
• The number 514291749 is divisible by 37 because: $749 + 291 + 514 = 1554$ which is a multiple of 37; or, alternatively, $749 = 20 \cdot 37 + 9$, $291 = 7 \cdot 37 + 32$, $514 = 13 \cdot 37 + 33$ and $9 + 32 + 33 = 74$ which is obviously a multiple of 37.

1.2 Algebra

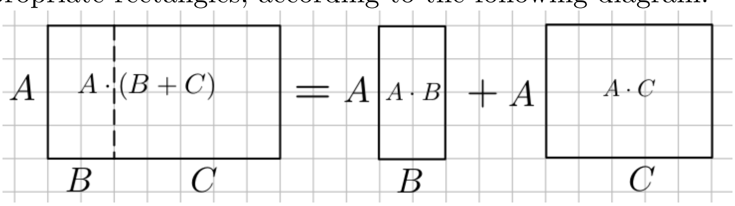
Definition 1.8 (mathematical expression) *A mathematical expression is a set of numbers or symbols that replace numbers, all combined together through mathematical operations.*

Simplifying an expression *means transforming the expression in such a way as to reduce the number of symbols or operations it contains, while preserving its overall numerical value.*

Property 1.9 (zero product property) *Two expressions A and B yield a product equal to zero if and only if at least one of them is zero, formally:* $A \cdot B \Longleftrightarrow A = 0 \lor B = 0$

Property 1.10 (distributive property) *Given the expressions A, B, C, the following equality generally holds:* $A \cdot (B + C) = A \cdot B + A \cdot C$
The above equality: read from left to right is properly called the "distributive property", while read from right to left it is called "factoring out" or "common factor extraction".

Proof. We can perform a geometric proof by associating the numerical values of the expressions A, B, C with the lengths of the sides of appropriate rectangles, according to the following diagram.



Definition 1.9 (power with natural exponent) *Let: $a \in \mathbb{R}$, $n \in \mathbb{N}$, we define "power with natural exponent" and read "a raised to n" the expression: $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$ if $n \neq 0$, while if $n = 0$ and $a \neq 0$ we define: $a^0 = 1$. a is called the "base", n is called the "exponent".*

Property 1.11 (power with natural exponent) *Let: $a, b \in \mathbb{R}$, $m, n \in \mathbb{N}$, we derive the following properties.*

- **Product of powers with the same base**

  $a^n \cdot a^m = a^{n+m}$

Proof. $a^n \cdot a^m = \underbrace{a \cdots a}_{n \text{ times}} \cdot \underbrace{a \cdots a}_{m \text{ times}} = \underbrace{a \cdots a \cdot a \cdots a}_{n+m \text{ times}} = a^{n+m}$

- **Quotient of powers with the same base**

  $\dfrac{a^n}{a^m} = a^{n-m}$ *if* $n \geq m$

Proof. If $n \geq m$: $\dfrac{a^n}{a^m} = \dfrac{\cancel{\overbrace{a \cdots a}^{m\ times}} \cdot \overbrace{a \cdots a}^{n-m\ times}}{\cancel{a^m}} = a^{n-m}$

- **Product of powers with the same exponent**

  $a^n \cdot b^n = (a \cdot b)^n$

Proof. $a^n \cdot b^n = \underbrace{a \cdots a}_{n\ times} \cdot \underbrace{b \cdots b}_{n\ times} = \underbrace{(a \cdot b) \cdots (a \cdot b)}_{n\ times} = (a \cdot b)^n$

- **Quotient of powers with the same exponent**

  $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$

Proof. $\dfrac{a^n}{b^n} = \dfrac{\overbrace{a \cdots a}^{n\ times}}{\underbrace{b \cdots b}_{n\ times}} = \underbrace{\dfrac{a}{b} \cdots \dfrac{a}{b}}_{n\ times} = \left(\dfrac{a}{b}\right)^n$

- **Power of a power**

  $(a^n)^m = a^{n \cdot m}$

Proof. $(a^n)^m = \underbrace{a^n \cdots a^n}_{m\ times} = \underbrace{\overbrace{(a \cdots a)}^{n\ times} \cdots \overbrace{(a \cdots a)}^{n\ times}}_{n \cdot m\ times} = a^{n \cdot m}$

Definition 1.10 (power with integer exponent) *Let:* $a \in \mathbb{R}$, $n \in \mathbb{Z}$, *we define:* $a^{-n} = \dfrac{1}{a^n}$

Definition 1.11 (root of a real number) *Let:* $a \in \mathbb{R}$, $n \in \mathbb{N}$, $n \neq 0$. *If there exists* $b \in \mathbb{R}$ *such that* $b^n = a$, *then* b *is called the "n-th root" of* a, *formally:* $\sqrt[n]{a} = b \overset{def}{\Longleftrightarrow} a = b^n$
a is called the "radicand", n is called the "index" of the root.
If $n = 2$ the index may be omitted.

$\boxed{\textbf{Property}}$ **1.12 (root of a real number)** *The following equalities hold, assuming all expressions are defined.*

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}; \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}; \quad \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m;$$

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[n \cdot m]{a}; \quad \sqrt{a^2} = |a|$$

Definition 1.12 (power with rational exponent) *Let:* $a > 0$; $n, m \in \mathbb{N}$. *We define:* $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Definition 1.13 (power with real exponent) *Let:*

$a > 0, x \in \mathbb{R}$,

$A = \{a^q \in \mathbb{R} | \, q \in \mathbb{Q}, q \leq x\}$, $B = \{a^q \in \mathbb{R} | \, q \in \mathbb{Q}, q \geq x\}$

We define: $a^x =$ *the separating element between A and B.*

$= \log_a(a^{\log_a x+\log_a y}) \stackrel{\text{def}}{=} \log_a x + \log_a y$

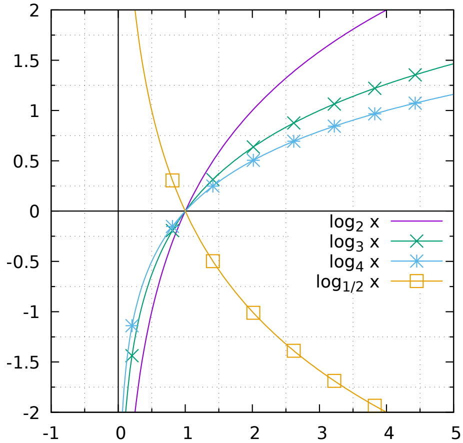
(4) $\log_a\left(\dfrac{x}{y}\right) = \log_a(x \cdot y^{-1}) \stackrel{3}{=} \log_a x + \log_a(y^{-1}) \stackrel{2}{=} \log_a x - \log_a y$

(5) $\log_b x \stackrel{1}{=} \log_b a^{\log_a x} \stackrel{2}{=} (\log_a x) \cdot \log_b a \Rightarrow \log_a x = \dfrac{\log_b x}{\log_b a}$ ■

Definition 1.80 (logarithmic function) *A logarithmic function is any function of the form:* $y = \log_a x$.

Property 1.73 (logarithmic function: increasing/decreasing) .

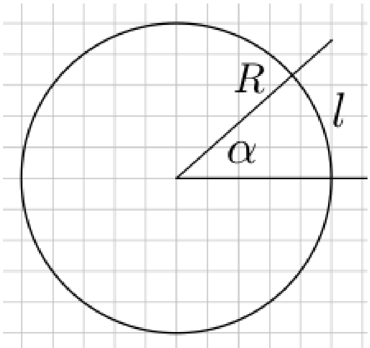
The logarithmic function $y = \log_a x$ *is strictly increasing or strictly decreasing if respectively* $a > 1$ *or* $0 < a < 1$.



1.14 Trigonometric Functions

Definition 1.81 (radian) *The measure of an angle* α *in radians, symbol rad, is defined as the ratio between the length* l *of the arc corresponding to* α*, taken on any circle of radius* R *centered at the vertex of the angle, and the length of that radius; formally:*

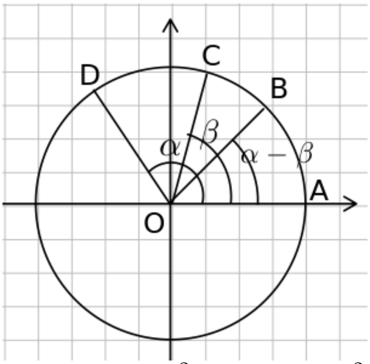
$$\alpha = \frac{l}{R} \ rad$$



Conversion between degrees and radians can be done by considering that:

$\dfrac{\alpha(\circ)}{\alpha(rad)} = \dfrac{180}{\pi}$, so:

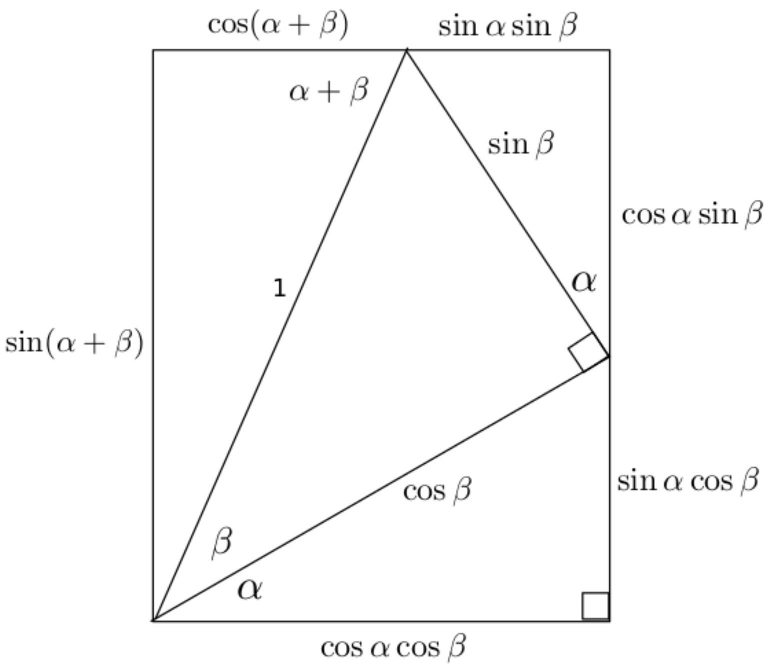
$\alpha(\circ) = \alpha(rad)\dfrac{180}{\pi}; \quad \alpha(rad) = \alpha(\circ)\dfrac{\pi}{180}$



one can start from the diagram on the unit circle shown in the figure, where
$\alpha = \widehat{AOD}$
$\beta = \widehat{AOC}$
$\alpha - \beta = \widehat{COD} = \widehat{AOB}$
The last equality implies that
$\overline{CD} = \overline{AB} \Rightarrow \overline{CD}^2 = \overline{AB}^2$
that is:
$(x_C - x_D)^2 + (y_C - y_D)^2 = (x_A - x_B)^2 + (y_A - y_B)^2$
$(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 = (1 - \cos(\alpha-\beta))^2 + (0 - \sin(\alpha-\beta))^2$
$\cos^2\beta + \cos^2\alpha - 2\cos\beta\cos\alpha + \sin^2\beta + \sin^2\alpha - 2\sin\beta\sin\alpha =$
$= 1 + \cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + \sin^2(\alpha-\beta)$
from which
$2 - 2\cos\beta\cos\alpha - 2\sin\beta\sin\alpha = 2 - 2\cos(\alpha-\beta)$
and finally
$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

The other addition and subtraction formulas can be easily derived from general properties of trigonometric functions.

Alternatively, if α and β are between 0 and 90°, all the addition and subtraction formulas can be derived from the following ingenious geometric diagram. ■



Property **1.83 (Heron's formula)** *The area of a triangle with sides a, b, c is:*

$$A = \sqrt{p(p-a)(p-b)(p-c)}$$

where p is the semiperimeter.

Proof. This follows from Brahmagupta's formula by considering one side to have zero length. ■

1.16 Vectors

Definition **1.85 (vector)** *A vector is any quantity that can be represented by a directed segment.*

Graphically, a vector can therefore be represented by an arrow, defined by three characteristics:
1. Magnitude = length of the segment
2. Line of action = line on which the segment lies or a parallel to it
3. Direction = orientation of the segment (starting point and endpoint)

According to Anglo-Saxon convention:

$$\text{Line of action} + \text{Direction} = \text{Direction}$$

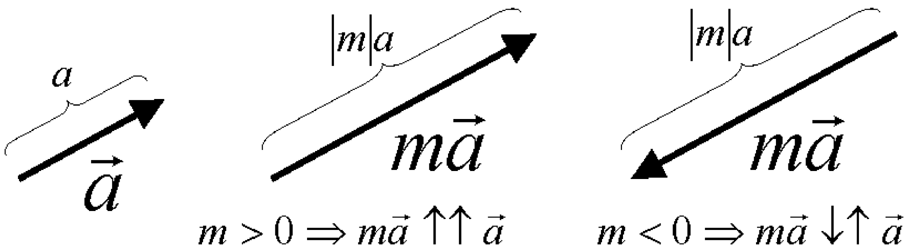
Algebraically, vectors can be represented by letters with an arrow above, e.g. $\vec{a}$, or in bold $\mathbf{a}$, or if the initial point is A and the endpoint is B, then $\overrightarrow{AB}$.

Symbolic conventions:

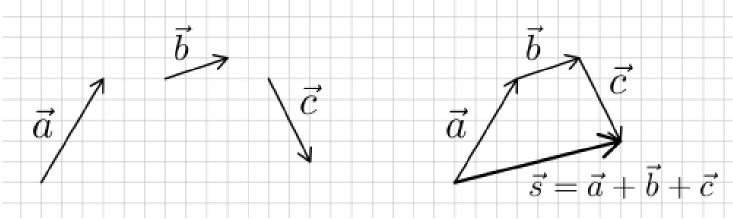
Magnitude of $\vec{a} = |\vec{a}| = a$;

Magnitude of $\overrightarrow{AB} = |\overrightarrow{AB}| = \overline{AB}$

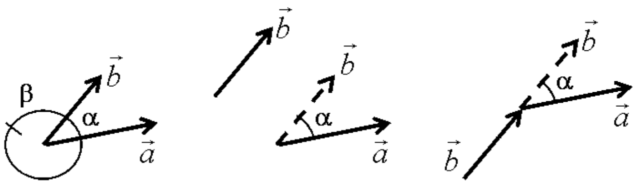
Definition **1.86 (scalar multiplication)** *The multiplication of a number m by a vector $\vec{a}$ gives the vector $m\vec{a}$ having:*
magnitude = magnitude of $\vec{a}$ times the absolute value of m;
line of action = line of action of $\vec{a}$;
direction = same or opposite to $\vec{a}$ depending on whether $m > 0$ or $m < 0$



Definition 1.87 (vector sum (tip-to-tail method)) . *The sum of multiple vectors, previously ordered and translated so that the tip of one coincides with the tail of the next, is the vector whose initial point coincides with the tail of the first and final point with the tip of the last.*



Definition 1.88 (included angle) . *The included angle between two vectors is the smaller angle formed by the two vectors, after suitably translating one of them so that their initial points coincide.*



$\alpha =$ *included angle;* $\beta =$ *non-included angle*

Definition 1.89 (dot product) *It is the product of two vectors* $\vec{a}$ *and* $\vec{b}$*, denoted by the symbol* $\vec{a} \cdot \vec{b}$*, and yields a scalar defined as the product of their magnitudes and the cosine of the included angle* α*.*

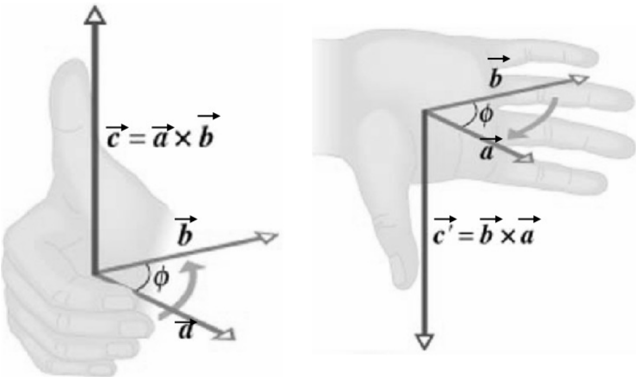
$$\vec{a} \cdot \vec{b} = ab \cos \alpha$$

Definition 1.90 (cross product) *It is the product of two vectors* $\vec{a}$ *and* $\vec{b}$*, denoted by the symbol* $\vec{a} \times \vec{b}$*, and yields a vector with:*

1. *magnitude* $= ab \sin \alpha$*, where* α *is the included angle;*

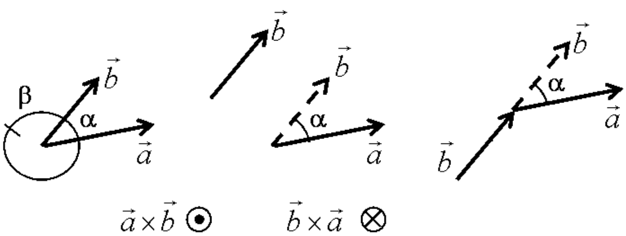
2. *direction perpendicular to the plane determined by the two vectors;*

3. *orientation given by the right-hand rule: with the palm of the right hand open and the thumb at* $90°$ *to the other fingers, imagine rotating the first vector toward the second one through the included angle; the thumb will point in the direction of the cross product vector.*



From the definition of cross product it immediately follows that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$, that is, the cross product is said to be *anticommutative.*

Definition 1.91 (vectors entering and exiting the page) *If the vectors $\vec{a}$ and $\vec{b}$ lie in the plane of the page, then to represent their cross product $\vec{a} \times \vec{b}$, which is a vector perpendicular to that plane, the symbols $\odot$ and $\otimes$ are used to indicate vectors perpendicular to the page pointing outward and inward, respectively.*



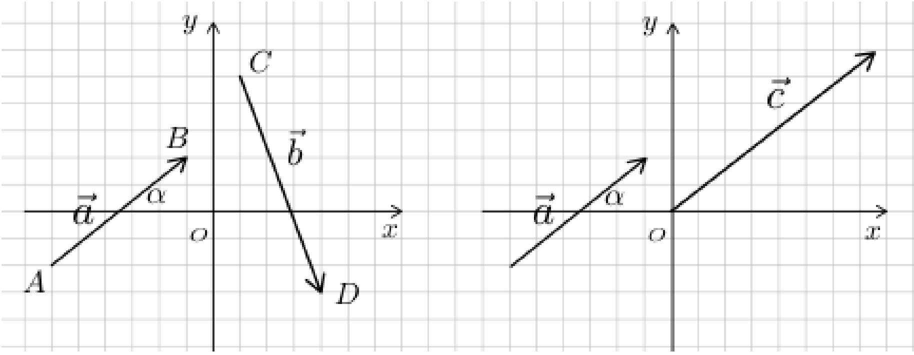
Property 1.84 (properties of operations with vectors) *Let m, n be scalars and $\vec{a}$, $\vec{b}$, $\vec{c}$ be vectors; the following properties can be demonstrated.*

Scalar-vector multiplication	*Dot product*	*Cross product*
$(m+n)\vec{a} = m\vec{a} + n\vec{a}$	$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$	$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
$(mn)\vec{a} = m(n\vec{a})$	$(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b})$	$(m\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b})$
$m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$	$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$	$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

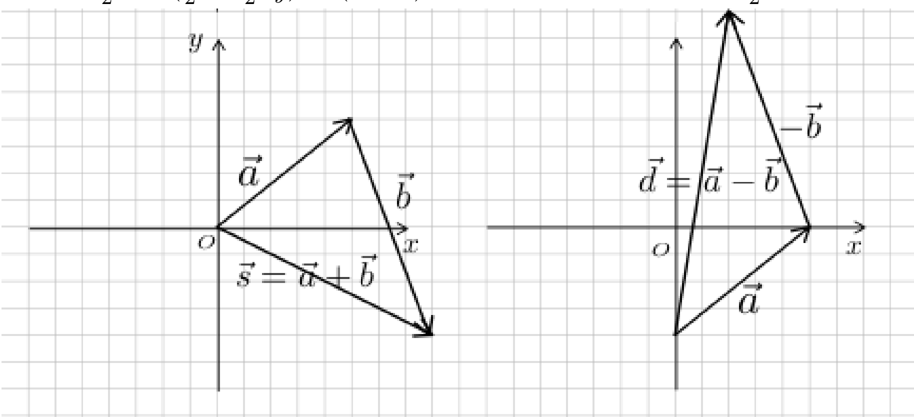
1.16.1 Cartesian representation of vectors

Definition 1.92 (unit vector of an oriented line) *The unit vector of an oriented line, also called a* **basis vector**, *is a vector of unit magnitude having the same direction and orientation as the line.*

By convention, the unit vectors, or **basis vectors**, *of the Cartesian axes x, y, z are denoted respectively by $\vec{i}, \vec{j}, \vec{k}$ or by $\hat{x}, \hat{y}, \hat{z}$.*



$\vec{a} = (-1+6, 2+2) = (5,4)$; $a = \sqrt{5^2+4^2} = 6.4$
$\vec{b} = (4-1, -3-5) = (3,-8)$; $a = \sqrt{3^2+(-8)^2} = 8.544$
$\tan\alpha = \frac{|a_x|}{|a_y|} = \frac{4}{5} = 0.8 \Rightarrow \alpha = \tan^{-1} 0.8 = 38.66°$
$\vec{c} = \frac{3}{2}\vec{a} = (\frac{3}{2}a_x, \frac{3}{2}a_y) = (7.5, 6)$; $c = \sqrt{7.5^2+6^2} = 9.6 = \frac{3}{2}a$



$\vec{s} = \vec{a} + \vec{b} = (5,4) + (3,-8) = (8,-4)$; $s = \sqrt{8^2+(-4)^2} = 8.94$
$\vec{d} = \vec{a} - \vec{b} = (5,4) - (3,-8) = (2,12)$; $d = \sqrt{2^2+12^2} = 12.166$

1.17 Inequalities and Sign Analysis

Definition 1.95 (inequality) *An inequality is a comparison between two expressions, called the "left-hand side" and the "right-hand side," depending on the values of certain letters in the expressions, called "unknowns."*

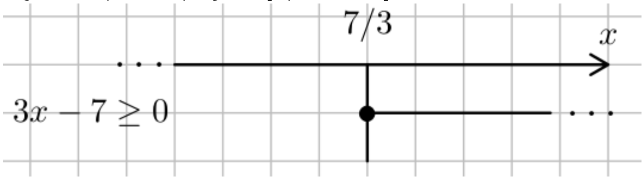
Solving an inequality *means finding the values of the unknowns that satisfy the inequality.*

The set of values that satisfy an inequality is called the "solution" of the inequality.

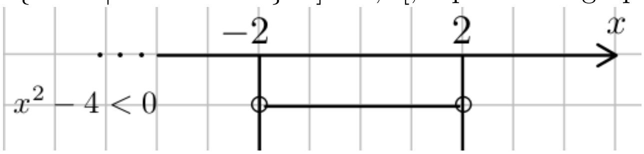
An inequality is said to be: **determined** *if it is satisfied for a (usually infinite) number of values of the unknowns;*

indeterminate *if it is satisfied for all admissible values of the unknowns;*
impossible *if it is not satisfied for any admissible value of the unknowns.*

Examples.
The solution of the inequality $3x - 7 \geq 0$ is $x \geq 7/3$, that is, the set $S = \{x \in \mathbb{R} | x \geq 7/3\} = [7/3, +\infty[$, represented graphically as:



The solution of the inequality $x^2 - 4 < 0$ is $-2 < x < 2$, that is, the set $S = \{x \in \mathbb{R} | -2 < x < 2\} =]-2, 2[$, represented graphically as:



Definition 1.96 (equivalent inequalities) *Two inequalities are said to be equivalent if they are satisfied by the same values of the unknowns, i.e., if they have the same solution set.*

Examples.
The inequalities: $x + 1 > 0$ and $3^x > 1/3$ are equivalent because they both have solution $x > -1$.
The inequalities: $9 - x^2 \geq 0$ and $(x + 3)(2^x - 8) \leq 0$ are equivalent because they both have solution $-3 \leq x \leq 3$, that is, the set $S = \{x \in \mathbb{R} | -3 \leq x \leq 3\} = [-3, 3]$.

***Postulate* 1.6 (First principle of inequality equivalence) .**
By adding the same expression to both sides of an inequality, one obtains an equivalent inequality.

***Postulate* 1.7 (Second principle of inequality equivalence) .**
By multiplying or dividing both sides of an inequality by the same non-zero expression, one obtains an equivalent inequality with: the same direction if the expression is positive, and reversed direction if the expression is negative.

Definition 1.97 (elementary inequalities) *An inequality is said to be elementary if it can be solved using only the two principles of equivalence.*

Examples. An elementary inequality can be, for example, the following one:

$2 - 5x \leq 6 \qquad (1)$

Indeed, we can solve it as follows.
Add -2 to both sides (first criterion)

$-2 + 2 - 5x \leq -2 + 6$

$-5x \leq 4$

Divide both sides by -5 (second criterion), remembering to reverse the direction of the inequality, since -5 is negative

$x \geq -\frac{4}{5}$

which is the solution of (1).

<u>**Definition**</u> **1.98 (sign analysis)** *To study the sign of an expression involving a single variable, say x, means to determine for which values of x the expression is:*

positive (symbol $+$)
negative (symbol $-$)
zero (symbol $\bigcirc$)
undefined (symbol $\nexists$ or $///$)

Examples.

$$\begin{array}{ccccc} & & \frac{5}{2} & & \\ 2x-5 & --- & \bigcirc & +++ & \end{array} \qquad \begin{array}{cccccc} & & -2 & & 2 & \\ x^2-4 & +++ & \bigcirc & --- & \bigcirc & +++ \end{array}$$

$$\begin{array}{cccc} & & 3 & \\ \frac{2}{x-3} & --- & \nexists & +++ \end{array} \qquad \begin{array}{cccccc} & & -2 & & 2 & \\ \sqrt{x^2-4} & +++ & \bigcirc & /// & \bigcirc & +++ \end{array}$$

$$\begin{array}{cccccc} & & -2 & & 2 & \\ \frac{1}{\sqrt{x^2-4}} & +++ & \nexists & /// & \nexists & +++ \end{array} \qquad \begin{array}{cccccc} & 0 & & \pi & & 2\pi \\ \sin x & \bigcirc & +++ & \bigcirc & --- & \bigcirc \end{array}$$

$$\begin{array}{cccccc} & & 5 & & 6 & \\ \ln(x-5) & /// & \nexists & --- & \bigcirc & +++ \end{array} \qquad \begin{array}{cccccc} & 0 & & \pi & & 2\pi \\ \dfrac{1}{\sin x} & \nexists & +++ & \nexists & --- & \nexists \end{array}$$

$\boxed{\textbf{Property}}$ **1.87 (sign of a quadratic polynomial)** *Known as the D.I.C.O. rule (Different Inside, Concordant Outside): the sign of a second-degree polynomial $p(x)$ is "different" from the sign of the leading coefficient for values of x *inside* the interval between the two roots, and "concordant" (same sign) for values of x *outside* that interval.*

A more detailed version of the above property is the following.